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EFFICIENT REANALYSIS OF LOCALLY MODIFIED STRUCTURES.(U)

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EFFICIENT REANALYSIS OF  
LOCALLY MODIFIED STRUCTURES

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## ABSTRACT

A unified efficient formulation for the static, dynamic, and stability reanalysis of locally modified structures is presented. The reanalysis problem is to find the structural responses when some of the element properties are adjusted as a result of design modifications. The reanalysis problem is formulated as a problem of much lower order than the original system. This is achieved by utilizing the linearity property of the structure using the pseudo-load concept, together with the solution of the original system. The modifications to the structure are treated as displacement dependent pseudo-loads of the system. By expressing the modified system response as linear combinations of the response of the original system and a term depending on the pseudo-load, a reduced set of response equations can be obtained. In the static and sinusoidal steady state analyses, this leads to a set of linear algebraic equations. For free vibration and stability analyses, this results in an eigenvalue problem. General modifications involve multiple parameters so that numerical solutions are required. Numerical examples are included. For the special case of a single parameter variation, a simple closed form formulation of the reanalysis problem is provided. Explicit reanalysis formulas are presented for changes in the cross-sectional area of truss members, the stiffness of beams, the thickness of plane stress elements, and lumped springs and masses.

## INTRODUCTION

Finite element analysis is an emerging technique which is used in the design of complex structural systems. From spacecraft to ground vehicles, from the largest dam to a small welded joint, almost all engineering structures have been analyzed in the design stage by the finite element method. The versatility of the finite element method has made possible the analysis of complex structural systems. The method provides the capability of modeling in great detail the real system under investigation.

As with many things in life, this increase in analytical capability also has a price tag: the cost of setting up and executing a finite element analysis and the loss of insight into the variations in the structural response when certain design parameters are changed. Both of these obstacles are related. It is these problems that are addressed in this paper.



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The cost problem arises because in the application of finite element analysis to design many configurations are investigated before the design is finalized. These investigations are a natural consequence of the trial and error process of current design procedures. Also, to obtain a parametric study of structural responses, the structure must be analyzed many times by varying one or more design parameters. Thus, to reduce the cost of finite element analyses, it is desirable to have a technique which will analyze the effects of system modifications efficiently and economically, especially when only a small portion of the original system is modified.

Many authors have treated the reanalysis problem. Reviews of reanalysis techniques can be found in Wang, Palazzolo, and Pilkey (1) and in Arora (2). In particular, the static reanalysis problem has been investigated by Argyris, Bronlund, and Roy (3), Kavlíe and Powell (4), and Noor and Lowder (5), among many others (1), (2). The eigenvalue or free vibration reanalysis problem has been studied by Weissenburger (6) (7), Stetson and Palma (8), Hallquist (9), and Hirai, Yoshimura, and Takamura (10). The forced sinusoidal response as a function of structural parameters has been studied by Done and Hughes (11) and verified experimentally by Done, Hughes, and Webby (12).

The purpose of this paper is to present a unified formulation for the static, free vibration, and forced sinusoidal response analysis of a locally modified structural system. The structure is assumed to be modeled by finite elements. The reanalysis problem is to find the structural responses when some of the element properties are changed as a result of design modifications. Because the modifications are localized, the reanalysis problem can be formulated as a problem of much lower order than the original system. This formulation is achieved by exploiting the linearity property of the structure using the pseudo-load concept, together with the solution of the original system. The modifications to the structure are treated as displacement dependent pseudo-loads of the system. Depending upon the type of analysis, these displacements ( $\bar{q}$ ) may be parts of the static displacements or the natural vibration mode shapes of the system. By expressing the modified system response as linear combinations of the responses of the original system and a term depending on the pseudo load (and hence on  $\bar{q}$ ), a reduced set of equations with  $\bar{q}$  as the unknown can be obtained. In the static and sinusoidal steady state analyses, this procedure leads to a set of linear algebraic equations in  $\bar{q}$ . For free vibration analyses, the procedure results in an eigenvalue problem in the form of an  $r^{\text{th}}$  order determinant, where  $r$  is the number of degrees of freedom affected by the modification. For this case, if all the modes are used in the formulation, the resulting equation is the exact characteristic equation of the modified system. Otherwise, an approximate formulation is obtained. Methods of improving solution accuracy when only a partial set of modes is available will be discussed. General modifications involve multiple parameters so that numerical solutions are required. For the special case of a single parameter variation, a simple closed form formulation of the reanalysis problem can be derived. A few explicit reanalysis formulas are presented in the following sections.

# GENERAL REANALYSIS PROBLEM: PSEUDO-LOAD FORMULATION

Consider a general linear structural system which can be described by the following equation

$$\underline{D} \underline{u} = \underline{f} \quad (1)$$

where  $\underline{D}$  is the generalized stiffness matrix. For a static problem,

$$\underline{D} = \underline{K}$$

For a steady state sinusoidal problem

$$\underline{D} = \underline{K} - \omega^2 \underline{M}$$

Suppose now the system is modified so that

$$\underline{D}' = \underline{D} + \underline{\Delta D} \quad (2)$$

Substitute (2) into (1) and rearrange

$$\underline{D} \underline{u}' = \underline{f} - \underline{\Delta D} \underline{u}' \quad (3)$$

where  $\underline{u}'$  is the solution of the modified system.

Premultiply (3) by  $\underline{Z}$ , the inverse of  $\underline{D}$ ,

$$\underline{u}' = \underline{Z} \underline{f} - \underline{Z} \underline{\Delta D} \underline{u}' \quad (4)$$

For local modifications,  $\underline{\Delta D}$  will be a highly sparse matrix; that is,  $\underline{\Delta D}$  contains only a small number of nonzero columns and rows. Let

$\hat{\underline{\Delta D}}$  = a submatrix of  $\underline{\Delta D}$  which contains only the nonzero columns of  $\underline{\Delta D}$

Furthermore, let the nonzero column of  $\underline{\Delta D}$  be the  $j_1, j_2, \dots, j_r$  columns, and let

$$\hat{\underline{u}}' = \begin{pmatrix} u_{j_1} \\ u_{j_2} \\ \vdots \\ u_{j_r} \end{pmatrix}$$

$\hat{\underline{u}}'_s$  = the column vector formed by deleting  $\hat{\underline{u}}'$  from  $\underline{u}'$

Similar definitions apply to  $\underline{g}$  and  $\hat{\underline{u}}_s$ . With these definitions, (4) can be simplified to

$$\underline{u}' = \underline{u} - \underline{Z} \hat{\underline{\Delta D}} \hat{\underline{u}}' \quad (5)$$

Now, define

$$\underline{Y}_i = \underline{Z} \hat{\underline{\Delta D}}_i \quad (6a)$$

or

$$\underline{Y} = \underline{Z} \hat{\underline{\Delta D}} \quad (6b)$$

where  $\hat{\underline{\Delta D}}_i$  =  $i^{\text{th}}$  column of  $\hat{\underline{\Delta D}}$ . Equation (5) can be written as

$$\underline{u}' = \underline{u} - \underline{Y} \hat{\underline{u}}' \quad (7)$$

Rearrange the equations in (2) to get the following partitioned set of equations.

$$\begin{Bmatrix} \hat{\underline{u}}' \\ \hat{\underline{u}}'_s \end{Bmatrix} = \begin{Bmatrix} \hat{\underline{u}} \\ \hat{\underline{u}}_s \end{Bmatrix} - \begin{Bmatrix} \hat{\underline{Y}} \\ \hat{\underline{Y}}_s \end{Bmatrix} (\hat{\underline{u}}') \quad (8)$$

From the first  $r$  equations of (8),

$$\hat{\underline{u}}' = \hat{\underline{u}} - \hat{\underline{Y}} \hat{\underline{u}}'$$

or

$$\hat{\underline{u}}' = (\underline{I} + \hat{\underline{Y}})^{-1} \hat{\underline{u}} \quad (9)$$

and  $\hat{\underline{u}}'_s$  can be computed from the lower portion of equation (8),

$$\hat{\underline{u}}'_s = \hat{\underline{u}}_s - \hat{\underline{Y}}_s \hat{\underline{u}}' \quad (10)$$

Note that equations (9) and (10) provide the solution of the modified problem. The computational efficiency obtained in solving the modified system formulation stems from the fact that equation (2),

which is a system of  $r$  equations, is to be solved instead of solving  $n$  equations, which would be the case if the modified system were to be solved directly. The additional information, i.e., matrix  $Y$ , needed in this formulation can be obtained by solving the original system with  $r$  different loadings. This solution is accomplished by forward-backward substitution once the original general stiffness matrix is factored into upper ( $U$ ) and lower ( $L$ ) triangular forms.

Note that in the above derivation we used only the linearity property of the system. Alternatively, it can be shown that equations (9) and (10) can be derived directly from the Woodbury formula of numerical analysis, which is also known as the matrix inversion lemma in the control theory literature.

In the following sections, equations (9) and (10) will be applied to the reanalysis of static response, steady state sinusoidal response, and free vibration problems.

#### STATIC REANALYSIS

For a static problem

$$\underline{D} = \underline{K} \quad (11)$$

and  $\underline{Z} = \underline{K}^{-1}$  is the system flexibility. Since

$$\Delta \underline{D} = \Delta \underline{K} \quad (12)$$

equation (6a) becomes

$$Y_i = \underline{K}^{-1} \hat{\Delta K}_i \quad (13)$$

where  $\hat{\Delta K}_i$  is the  $i^{\text{th}}$  nonzero column of  $\Delta K$ . Assuming the original system is solved, then  $\underline{u}$  is known, and  $\underline{K}$  can be written

$$\underline{K} = \underline{L} \underline{U} \quad (14)$$

In terms of upper and lower triangular forms, equation (1) becomes

$$\underline{U} \underline{z} = \underline{f} \quad (15)$$

$$\underline{L} \underline{u} = \underline{z} \quad (16)$$

The reanalysis procedure is

1. Formulate the modification in the stiffness matrix as

$$\Delta K = \sum_{i=1}^n \Delta K_{e_i} \quad (17)$$

where  $\Delta K_{e_i}$  = the contribution in the global stiffness matrix due to the modification of the  $i^{\text{th}}$  element stiffness characteristics.

2. Compute matrix  $\underline{Y}$  by the following algorithm. The  $i^{\text{th}}$  column is computed as

$$\underline{U} \underline{Z}_i = \hat{\Delta K}_i \quad (18)$$

$$\underline{L} \underline{Y}_i = \underline{Z}_i \quad (19)$$

where  $\hat{\Delta K}_i = i^{\text{th}}$  nonzero column of  $\Delta K$ . In equation (18),

$\underline{Z}_i$  is computed by backward substitution; and from (19)  $\underline{Y}_i$  is computed by forward substitution. Carry out these procedures in (18), (19) for  $i = 1, 2, \dots, r$ .

3. Form matrix  $\hat{\underline{Y}}$  by selecting the  $j_1^{\text{th}}, j_2^{\text{th}}, \dots, j_r^{\text{th}}$  row of  $\underline{Y}$ .

4. Compute  $\underline{W}$  from

$$\underline{W} \hat{\underline{U}}' = \hat{\underline{U}} \quad (20)$$

where

$$\underline{W} = \underline{I} + \underline{Y} \quad (21)$$

Equation (20) can be solved by decomposition:

$$\underline{W} = \underline{L}_w \underline{U}_w \quad (22a)$$

$$\underline{U}_w = \hat{\underline{Z}} = \hat{\underline{U}} \quad (22b)$$

$$\underline{L}_w \hat{\underline{U}}' = \hat{\underline{Z}} \quad (22c)$$

5. Compute  $u'$  from equation (7).

This completes the solution of the modified system. Note that in the final step,  $q'$  will be computed again. The results can be used to check against  $q'$  obtained from the solution of equation (20). Note also that in this computational procedure, the matrix partition indicated in equation (8) does not need to be carried out explicitly.

For general modifications involving many degrees of freedom (dof), the above procedure can be programmed as a post-processor of any general finite element code. For simple modifications involving few degrees of freedom, the reanalysis equations can be formulated manually from finite element solution outputs. It is also possible to develop a "closed form" formula for simple local modifications.

To illustrate the development of an analytical modification expression, consider the case of adding spring  $\Delta k$  between dof  $j_1$  and dof  $j_2$  in a general finite element system.

Step 1: For this case, the first step of the procedure is readily accomplished since it is known that

$$\Delta K = \begin{bmatrix} 0 & & & 0 \\ & \Delta k & -\Delta k & \\ & -\Delta k & \Delta k & \\ 0 & & & 0 \end{bmatrix} \begin{matrix} \\ j_1 \\ j_2 \\ \end{matrix} \quad (23)$$

That is,  $\Delta K$  has only four nonzero elements. It follows that

$$\hat{\Delta K}_1 = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ +\Delta k \\ 0 \\ \vdots \\ 0 \\ -\Delta k \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \quad \text{is the first nonzero column of } \Delta K, \text{ and}$$

$$\widehat{\Delta K}_2 = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ -\Delta k \\ 0 \\ \vdots \\ 0 \\ +\Delta k \\ 0 \\ \vdots \\ 0 \end{Bmatrix}$$

is the second nonzero column of  $\Delta K$ .

These can be expressed as

$$\widehat{\Delta K}_1 = \Delta k \underline{P} \quad (24a)$$

$$\widehat{\Delta K}_2 = -\Delta k \underline{P} \quad (24b)$$

where

$$\underline{P} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \quad \begin{matrix} j_1^{\text{th}} \text{ row} \\ \\ \\ j_2^{\text{th}} \text{ row} \end{matrix} \quad (24c)$$

Step 2: Compute matrix  $\underline{Y}$  using

$$\underline{K} \underline{Y}_1 = \hat{\Delta \underline{K}}_1 = \Delta k \underline{P} \quad (25a)$$

$$\underline{K} \underline{Y}_2 = \hat{\Delta \underline{K}}_2 = -\Delta k \underline{P} \quad (25b)$$

For this special case, it is sufficient to solve

$$\underline{K} \bar{\underline{Y}}_1 = \underline{P} \quad (26)$$

This can be accomplished using the procedure of equations (18) and (19) after obtaining  $\underline{Y}_1$ ,

$$\underline{Y}_1 = \Delta k \bar{\underline{Y}}_1$$

$$\underline{Y}_2 = -\Delta k \bar{\underline{Y}}_2$$

and

$$\underline{Y} = [\underline{Y}_1 \ \underline{Y}_2] = \Delta k [\bar{\underline{Y}}_1 \ -\bar{\underline{Y}}_2] \quad (27)$$

Step 3: Form  $\hat{\underline{Y}}$ , which is the matrix formed by the  $j_1^{\text{th}}$  and  $j_2^{\text{th}}$  row of  $\underline{Y}$ . Hence

$$\hat{\underline{Y}} = \Delta k \begin{bmatrix} \bar{Y}_{j_1 1} & -\bar{Y}_{j_1 2} \\ \bar{Y}_{j_2 1} & -\bar{Y}_{j_2 2} \end{bmatrix} \quad (28)$$

Step 4: To find  $\hat{\underline{Y}}^{-1}$ , first form  $\underline{W}$  as

$$\underline{W} = \underline{I} + \underline{Y} = \begin{bmatrix} 1 + b & -b \\ a & 1 - a \end{bmatrix} \quad (29)$$

where

$$a = \Delta k \bar{y}_{j_2 1}$$

(30)

$$b = \Delta k \bar{y}_{j_1 1}$$

Hence

$$\hat{\underline{u}}' = \underline{w}^{-1} \hat{\underline{u}}$$

which can be expressed explicitly as

$$\hat{\underline{u}}' = \frac{1}{C} \begin{Bmatrix} (1-a)u_{j_1} + bu_{j_2} \\ -au_{j_1} + (1+b)u_{j_2} \end{Bmatrix} \quad (31)$$

where  $C = 1 + b - a$ . From (30) and (31)

$$\hat{\underline{u}}' = \frac{1}{C} \begin{Bmatrix} (1-\Delta k \bar{y}_{j_2 1}) u_{j_1} + \Delta k \bar{y}_{j_1 1} u_{j_2} \\ -\Delta k \bar{y}_{j_2 1} u_{j_1} + (1 + \Delta k \bar{y}_{j_2 1}) u_{j_2} \end{Bmatrix} \quad (32)$$

$$C = 1 + (\bar{y}_{j_1 1} - \bar{y}_{j_2 1}) \Delta k$$

Note that for this case, we have obtained  $\hat{u}$  (and hence  $y'$ ) explicitly as a function of  $\Delta k$ , the varying parameter. As a consequence, we have solved a family of problems, as well as a particular reanalysis problem, since for any value of  $\Delta k$ , the response of the modified system can be computed by simple calculations using equations (32) and (7).

As a numerical example, consider the simple truss shown in Fig. 1. Suppose now that the cross-sectional area of member 4 is changed to  $A'_4 = (\sqrt{2}/2)(1+\alpha)$ . We want to find the response of the modified system. The original system stiffness is

$$\underline{K} = 0.125(10^6) \begin{bmatrix} 5 & 1 & 0 & 0 \\ 1 & 5 & 0 & -4 \\ 0 & 0 & 5 & -1 \\ 0 & -4 & -1 & 5 \end{bmatrix}$$

and the decomposed matrices are

$$\underline{K} = \underline{L} \underline{U}$$

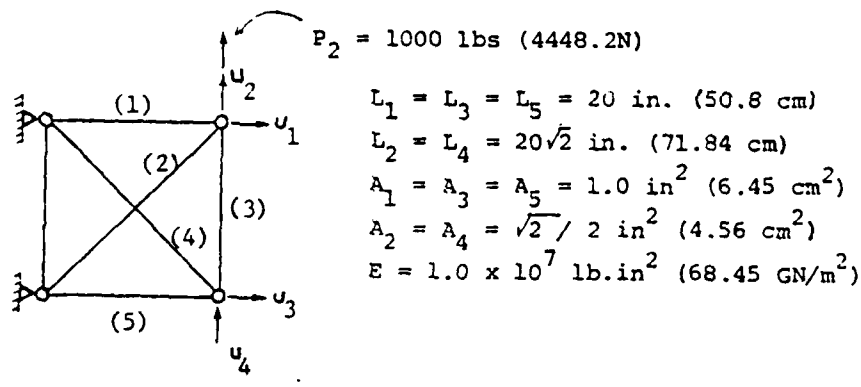


Fig. 1 Truss used as an example problem

where

$$\underline{L} = 0.125(10^6) \begin{bmatrix} \sqrt{5} & 0 & 0 & 0 \\ 1\sqrt{5} & \sqrt{24}/5 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 \\ 0 & -\sqrt{3}/10 & -1/\sqrt{5} & \sqrt{22}/5 \end{bmatrix}$$

$$\underline{u} = \frac{1}{0.125(10^6)} \underline{L}^T$$

The loading vector is

$$\underline{f} = \begin{Bmatrix} 0 \\ 1000 \\ 0 \\ 0 \end{Bmatrix}$$

The displacements of the original system are calculated to be

$$\underline{u} = (2/11)10^{-3} \begin{Bmatrix} 6 \\ 30 \\ 5 \\ 25 \end{Bmatrix}$$

For the modified system, it can be shown that

$$\Delta K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & e & -e \\ 0 & 0 & -e & e \end{bmatrix}$$

where  $e = 0.125(10^6)\alpha$ . Thus, the matrix  $\underline{Y}$  can be computed to be

$$\underline{Y} = (\alpha/11) \begin{bmatrix} 1 & -1 \\ -5 & 5 \\ 1 & -1 \\ -6 & 6 \end{bmatrix}$$

Now, to use equation (32), we have  $j_1 = 3$ ,  $j_2 = 4$  and calculate

$$u_{j_1} = u_3 = (10/11)(10^{-3})$$

$$u_{j_2} = u_4 = (50/11)(10^{-3})$$

$$Y_{j_1 1} = 1$$

$$Y_{j_2 1} = -6$$

$$\Delta k = \alpha/11$$

From equation (32)

$$C = 1 + (\bar{Y}_{j_1 1} - \bar{Y}_{j_2 1})\Delta k = 1 + 7\alpha/11 \quad (33a)$$

$$u_3' = (1/C)[(1 + 6\alpha/11)u_3 + (\alpha/11)u_4] \quad (33b)$$

$$u_4' = (1/C)[6\alpha u_3/11 + (1 + \alpha/11)u_4] \quad (33c)$$

and

$$u_1' = u_1 - \alpha(u_3' - u_4')/11 \quad (33d)$$

$$u_2' = u_2 - 5\alpha(-u_3' + u_4')/11 \quad (33e)$$

Hence, the modified system solutions are found in terms of the original system solution and the design parameter.

Note that equations (33) offer an opportunity to select design parameter  $\alpha$  for prescribed static responses. For example, from equations (33a and b), we can solve for  $\alpha$  as

$$\alpha = 11(u_3 - u_3')/(7u_3' - 6u_3 - u_4) \quad (34)$$

Suppose we want the modified system to have a smaller deflection at dof 3. Say we want  $u_3' = 0.9 u_3$ . Then from equation (34), the required  $\alpha$  is

$$\alpha = 0.234$$

Thus, using the modification theory, we can perform a parametric study for simple design modifications, as well as design the system for prescribed responses by properly choosing the design parameter value.

Although only one special case of adding a spring between two dof has been treated analytically here, many other special cases can also be derived in a similar manner.

#### STEADY STATE RESPONSE REANALYSIS

For a steady state analysis, let the forcing function and the response be sinusoidal functions of time, i.e.,

$$\underline{f} = \underline{f} \sin \omega t \text{ and } \underline{u} = \underline{u} \sin \omega t$$

where  $\omega$  is the forcing function frequency. This applies for undamped systems. The general stiffness matrix is

$$\underline{D} = \underline{K} - \omega^2 \underline{M} \quad (35)$$

The modification will involve changes in both the mass and stiffness matrices. Thus

$$\Delta \underline{D} = \Delta \underline{K} - \omega^2 \Delta \underline{M} \quad (36)$$

If this frequency response problem is to be treated directly, the modified system can be solved using the pseudoload formulation with equations (35) and (36) for  $\underline{D}$  and  $\Delta \underline{D}$ . If the modal superposition method is used for the analysis, then the receptance matrix  $\underline{Z}$  can be represented using a modal expansion

$$\underline{Z} = \sum_{l=1}^p \frac{\underline{\phi}_l \underline{\phi}_l^T}{G_l (\omega_l^2 - \omega^2)} \quad (37)$$

where  $\omega_l$  is the frequency,  $\underline{\phi}_l$  is the mode shape, and  $G_l$  is the generalized mass of the  $l^{\text{th}}$  mode. If  $p = n$ , equation (37) is exact. If  $p < n$ , which is usually the case with the normal sinusoidal mode method, equation (37) gives approximate results. The results can be improved by isolating the static response in  $\underline{Z}$ . It can be shown that equation (37) can be written as

$$\underline{Z} = \sum_{\ell=1}^n \frac{\phi_{\ell} \phi_{\ell}^T}{G_{\ell} (\omega_{\ell}^2 - \omega^2)} = \sum_{\ell=1}^n \left[ \sum_{a=1}^{R-1} \frac{\phi_{\ell} \phi_{\ell}^T}{G_{\ell} \omega_{\ell}^2} \left( \frac{\omega^2}{\omega_{\ell}^2} \right)^a - \frac{\phi_{\ell} \phi_{\ell}^T}{G_{\ell} \omega_{\ell}^2} \left( \frac{\omega^2}{\omega_{\ell}^2} \right)^R \right] \quad (38)$$

The integer R is a higher mode factor. Normally, the higher the value of R, the better the accuracy and lower the computational efficiency. It can be shown that

$$\underline{K}^{-1} = \sum_{\ell=1}^n \frac{\phi_{\ell} \phi_{\ell}^{-1}}{G_{\ell} \omega_{\ell}^2} \quad (39)$$

$$\underline{K}^{-1} (\underline{M} \underline{K}^{-1})^a = \sum_{\ell=1}^n \frac{\phi_{\ell} \phi_{\ell}^T}{G_{\ell} (\omega_{\ell}^2)^{a+1}} \quad (40)$$

Rewrite (38) as

$$\underline{Z} = \sum_{a=0}^{R-1} \left[ \sum_{\ell=1}^n \frac{\phi_{\ell} \phi_{\ell}^T (\omega^2)^a}{G_{\ell} (\omega_{\ell}^2)^{a+1}} \right] - \sum_{\ell=1}^n \frac{\phi_{\ell} \phi_{\ell}^T}{G_{\ell} \omega_{\ell}^2} \left( \frac{\omega^2}{\omega_{\ell}^2} \right)^R \quad (41)$$

Using (40) and (41),

$$\underline{Z} = \sum_{a=0}^{R-1} (\omega^2)^a (\underline{K}^{-1}) (\underline{M} \underline{K}^{-1})^a - \sum_{\ell=1}^n \frac{\phi_{\ell} \phi_{\ell}^T}{G_{\ell} \omega_{\ell}^2} \left( \frac{\omega^2}{\omega_{\ell}^2} \right)^R \quad (42)$$

The first series on the right hand side of (42) can be found from the static solution and is independent of the number of modes available. Hence, the improved receptance formula when only p modes are available is

$$\underline{Z} = \sum_{a=0}^{R-1} (\underline{K}^{-1}) (\underline{M} \underline{K}^{-1})^a - \sum_{\ell=1}^p \frac{\phi_{\ell} \phi_{\ell}^T}{G_{\ell} \omega_{\ell}^2} \left( \frac{\omega^2}{\omega_{\ell}^2} \right)^R \quad (43)$$

To find  $\underline{u}$  and  $\underline{Z}$ , proceed in the following manner.

$$\underline{u} = \underline{Z} \underline{f} = \sum_{a=0}^{R-1} \underline{u}_a - \sum_{l=1}^P \frac{\phi_l \phi_l^T \underline{f}}{G_l \omega_l^2} \left( \frac{\omega^2}{\omega_l^2} \right)^R \quad (44)$$

where

$$\underline{u}_a = \underline{Z}_a \underline{f} \quad (45)$$

$$\underline{Z}_a = \underline{K}^{-1} (\underline{M} \underline{K}^{-1})^a$$

Compute  $\underline{u}_0$  from

$$\underline{K} \underline{u}_0 = \underline{f} \quad (46a)$$

and calculate  $\underline{u}_a$  using

$$\underline{K} \underline{u}_a = \underline{M} \underline{u}_{(a-1)} \quad \text{for } a = 1 \text{ to } R - 1 \quad (46b)$$

This procedure can also be applied to find  $\underline{Y}$ . Once  $\underline{Y}$  is obtained, one can proceed as in the static reanalysis problem to find the response of the modified system using equations (2) and (10).

#### FREE VIBRATION REANALYSIS

If the external forcing function is zero, then  $\underline{u} = 0$  and equation (2) has a nontrivial solution if and only if

$$\det (\underline{I} + \hat{\underline{Y}}) = 0 \quad (47)$$

This is the frequency equation for free vibration of the modified system. For the modal formulation, the matrix  $\underline{Y}$  can be found using the formulation of the previous section. For free vibrations, equation (1) becomes

$$\underline{u}' = -\underline{Y} \hat{\underline{u}}' \quad (48)$$

By using the calculated modified system natural frequency  $\omega_j$  in  $\underline{Y}$  of Eq. (48), the system mode shape can be computed from (48) by properly assigning a unity value to one of the components of  $\underline{u}'$ . Any component of  $\underline{u}'$  will do as long as it is not a node point of that mode.

In the following, we will develop several special cases of frequency equations of modified systems. We will use the modal superposition approach assuming all modes are available and then indicate how to improve the results when only a partial set of modes are to be used.

Consider the special case of adding mass  $\Delta m$  at dof  $j$ . In general, a point mass in a three-dimensional space has 3 dof. Here it will be assumed that the mass is constrained for one-dimensional motion only. Neglect the contribution to the stiffness due to this added mass (or mass removal, if  $\Delta m$  is negative). Then

$$\Delta \underline{D} = -\omega^2 \Delta \underline{M} \quad (49)$$

where

$$\Delta \underline{M} = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ \vdots & & 0 & \Delta m_0 & \vdots \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

Thus

$$\hat{\Delta \underline{D}}_1 = -\omega^2 \left\{ \begin{matrix} 0 \\ \vdots \\ 0 \\ \Delta m \\ 0 \\ \vdots \\ 0 \end{matrix} \right\} = -\omega^2 \Delta m \underline{p} \quad (50)$$

with

$$\underline{P} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j_1 \quad (51)$$

Equation (5b) becomes

$$\underline{Y} = \underline{Z} \hat{\underline{\Delta D}}_1 = \omega^2 \Delta m \underline{Z} \underline{P} \quad (52)$$

Use Eq. (37) for  $\underline{Z}$ .

$$\underline{Y} = -\omega^2 \Delta m \sum_{\ell=1}^n \frac{\underline{\phi}_\ell \underline{\phi}_\ell^T \underline{P}}{G_\ell (\omega_\ell^2 - \omega^2)} \quad (53)$$

Since

$$\underline{\phi}_\ell^T \underline{P} = \phi_{j\ell} \quad (54)$$

equation (53) reduces to

$$\underline{Y} = -\omega^2 \Delta m \sum_{\ell=1}^n \frac{\underline{\phi}_\ell \phi_{j\ell}}{G_\ell (\omega_\ell^2 - \omega^2)} \quad (55)$$

Hence

$$\underline{\hat{Y}} = \underline{Y}_{j1} = -\omega^2 \Delta m \sum_{\ell=1}^n \frac{\phi_{j\ell}^2}{G_\ell (\omega_\ell^2 - \omega^2)} \quad (56)$$

From (56) and (47), the frequency equation for adding  $\Delta m$  at dof  $j_1$  is

$$1 - \omega^2 \Delta m \sum_{l=1}^n \frac{\phi_{j_1 l}^2}{G_l (\omega_l^2 - \omega^2)} = 0 \quad (57)$$

Let  $\omega = \omega_1'$  be the solution of equation (57). The mode shape of the modified system corresponding to natural frequency  $\omega_1'$  can be computed from

$$\phi_1' = Y \quad (\omega = \omega_1') \quad (58)$$

where  $Y$  is given by equation (53). In the above derivation, if the approximate  $Z$  defined by equation (43) is used in equation (52), then an approximate frequency equation is obtained.

Consider next the case of adding spring  $\Delta k$  between dof  $j_1$  and dof  $j_2$ . For this situation,  $\Delta D$  equals  $\Delta k$  of equation (23) so that

$$\begin{aligned} \hat{\Delta D}_1 &= \Delta k P \\ \hat{\Delta D}_2 &= -\Delta k P \end{aligned} \quad (59)$$

with  $P$  given by equation (24c). Thus,

$$Y_1 = Z \hat{\Delta D}_1 = \Delta k Z P = \Delta k \sum_{l=1}^n \frac{\phi_l \phi_l^T P}{G_l (\omega_l^2 - \omega^2)}$$

or

$$\underline{Y}_1 = \Delta k \sum_{l=1}^n \frac{\phi_l (\phi_{j_1 l} - \phi_{j_2 l})}{G_l (\omega_l^2 - \omega^2)} \quad (60)$$

Similarly

$$\underline{Y}_2 = \Delta k \sum_{l=1}^n \frac{\phi_l (-\phi_{j_1 l} + \phi_{j_2 l})}{G_l (\omega_l^2 - \omega^2)} \quad (61)$$

also

$$\hat{\underline{Y}} = \begin{bmatrix} Y_{j_1 1} & Y_{j_2 2} \\ Y_{j_2 1} & Y_{j_2 2} \end{bmatrix} \quad (62)$$

Hence the frequency equation is

$$\det (\underline{I} + \hat{\underline{Y}}) = 0$$

or

$$(1 + Y_{j_1 1})(1 + Y_{j_2 2}) - Y_{j_2 2} Y_{j_1 1} = 0 \quad (63)$$

From equations (6Q) and

$$Y_{j_1 1} = \Delta k \sum_{\ell=1}^n \frac{\phi_{j_1 \ell} (\phi_{j_1 \ell} - \phi_{j_2 \ell})}{G_{\ell} (\omega_{\ell}^2 - \omega^2)} \quad (64a)$$

$$Y_{j_1 2} = \Delta k \sum_{\ell=1}^n \frac{\phi_{j_1 \ell} (-\phi_{j_1 \ell} + \phi_{j_2 \ell})}{G_{\ell} (\omega_{\ell}^2 - \omega^2)} \quad (64b)$$

$$Y_{j_2 1} = \Delta k \sum_{\ell=1}^n \frac{\phi_{j_2 \ell} (\phi_{j_1 \ell} - \phi_{j_2 \ell})}{G_{\ell} (\omega_{\ell}^2 - \omega^2)} \quad (64c)$$

$$Y_{j_2 2} = \Delta k \sum_{\ell=1}^n \frac{\phi_{j_2 \ell} (-\phi_{j_1 \ell} + \phi_{j_2 \ell})}{G_{\ell} (\omega_{\ell}^2 - \omega^2)} \quad (64d)$$

Substitution of equations (64) into (63) leads to the frequency equation

$$1 + \Delta k \sum_{\ell=1}^n \frac{(\phi_{j_2 \ell} - \phi_{j_1 \ell})^2}{G_{\ell} (\omega_{\ell}^2 - \omega^2)} = 0 \quad (65)$$

Equation (65) is the frequency equation after adding spring  $\Delta k$  to the system. To find the mode shape of the modified system, choose  $\hat{u}_1 = 1.0$ , then

$$\hat{u}_2 = - \frac{1 + Y_{j_1 1}}{Y_{j_1 2}} \quad (66)$$

Having determined

$$\hat{u} = \begin{Bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{Bmatrix}$$

the mode shape can be computed using equation (1).

Consider now the special case of adding a spring between dof  $j_1$  and a fixed base. For this case,  $\phi_{j_2 l} = 0$  and the frequency equation reduces to

$$\sum_{l=1}^n \frac{\phi_{j_1 l}^2}{G_l (\omega_l^2 - \omega^2)} = \frac{1}{\Delta k} \quad (67)$$

Furthermore, let  $\Delta k \rightarrow \infty$ , that is, introduce a rigid constraint at dof  $j_1$ . The frequency equation becomes

$$\sum_{l=1}^n \frac{\phi_{j_1 l}^2}{G_l (\omega_l^2 - \omega^2)} = 0 \quad (68)$$

As a numerical example, consider the truss of Fig. 1 with unit mass at nodes A and B. Then for free vibration, the original system mass matrix is

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (69)$$

The corresponding stiffness matrix is

$$[K] = a \begin{bmatrix} 5 & 1 & 0 & 0 \\ 1 & 9 & 0 & -8 \\ 0 & 0 & 5 & -1 \\ 0 & -8 & -1 & 9 \end{bmatrix}$$

$$a = 0.125(10^6)$$

The natural frequencies are

$$\begin{aligned} \omega_1^2 &= 0.7639a & \omega_2^2 &= 4.764a \\ \omega_3^2 &= 5.236a & \omega_4^2 &= 9.236a \end{aligned} \quad (71)$$

with the corresponding mode shape vectors

$$\phi_1 = \begin{Bmatrix} 0.1625 \\ -0.6882 \\ -0.1625 \\ -0.6882 \end{Bmatrix} \quad \phi_2 = \begin{Bmatrix} 0.6882 \\ -0.1625 \\ 0.6882 \\ 0.1625 \end{Bmatrix} \quad \phi_3 = \begin{Bmatrix} 0.6882 \\ 0.1625 \\ -0.6882 \\ 0.1625 \end{Bmatrix} \quad \phi_4 = \begin{Bmatrix} 0.1625 \\ 0.6882 \\ 0.1625 \\ -0.6882 \end{Bmatrix} \quad (72)$$

Note that the mode shapes have been chosen to yield the unit generalized mass, or  $G_l = 1.0$  for  $l = 1$  to 4.

Consider the modification of doubling the cross-sectional area of member 3, that is, let  $\Delta A_3 = 1.0 \text{ in}^2$ . Since member 3 acts like a spring between dof 2 and 4, we can use equation (65) as the frequency equation for the modified system. Define  $a\bar{\omega}^2 = \omega^2$ ,  $a\bar{\omega}_l^2 = \omega_l^2$ . Use  $\Delta k = \Delta A_3 E/L_3$ ,  $j_1 = 2$ ,  $j_2 = 4$ . Equation (65) becomes

$$1 + \frac{\Delta A_3 E}{L_3} \sum_{\ell=1}^4 \frac{(\phi_{2\ell} - \phi_{4\ell})^2}{a(\bar{\omega}_\ell^2 - \bar{\omega}^2)} = 0 \quad (73)$$

Using the modal data, this frequency equation becomes

$$\frac{0.1056}{4.764 - \bar{\omega}^2} + \frac{1.8945}{9.236 - \bar{\omega}^2} = -0.25 \quad (74)$$

Equation (74) is the frequency equation of the modified system. It may appear odd that equation (74) has only two roots when the system has 4 dof. This condition results because in mode numbers 1 and 3 of the original system, there is no relative motion between dof 2 and 4, and hence the first and third natural frequencies will not be altered by the modification of the cross section of member 3. The solutions of equation (74) are the new frequencies of modes 2 and 4 of the original system. These solutions are

$$\bar{\omega}^2 = 4.917, 17.083$$

which are the same as in the direct solution of the modified system.

It is of interest that in equation (73), we can specify a natural frequency  $\bar{\omega}$  of the modified system and compute the required  $\Delta A_3$ . This procedure provides a means of designing for a specified natural frequency.

The general procedure set forth here can be used to derive frequency equations for other particular cases, such as a change in the EI of a beam element. This will be the subject of future presentations.

#### CONCLUDING REMARKS

In this paper, a unified formulation is presented for the efficient solution of reanalysis problems for locally modified systems. This technique is applicable to both static and dynamic (steady state and free vibration) reanalysis. For simple reanalyses involving one parameter, closed form formulas are derived and procedures of using these are outlined. These closed form formulas not only solve the desired reanalysis problem, but they also provide a means of performing rapid and economical parametric studies of structural systems. Furthermore, for one parameter problems, the reanalysis formulation can be inverted to solve the problem of selecting a parameter value which results in a given system response. For modifications involving more than one parameter, the formulation transforms the problem into one equal dimension to the number of

degrees of freedom affected by the modification. Thus, it would appear that this formulation can be programmed on a small desk top (or home) computer. The original system can be solved on a large computer and then a small computer can be used for computing, storing, and transferring the required data for reanalysis. The reanalysis can be performed economically and rapidly, if the turn-around time as well as computing time is considered, for many local modifications during the design process. Hopefully, this approach can also furnish more insight into the behavior of a complex structural system as a local parameter varies.

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